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Introductive Backgrounds to Modern Quantum Mathematics with Application to Nonlinear Dynamical Systems

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Abstract Introductive backgrounds to a new mathematical physics discipline—Quantum Mathematics—are discussed and analyzed both from historical and from analytical points of view. The magic properties of the second quantization method, invented by V. Fock in 1932, are demonstrated, and an impressive application to the nonlinear dynamical systems theory is considered.

The Authors devote their article to their Friend and Teacher academician Prof. Anatoliy M. Samoilenko on occasion of his 70 years-Birthday with great compliments and gratitude to his brilliant talent and impressive impact to modern theory of nonlinear dynamical systems of mathematical physics and nonlinear analysis

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1 Introduction

By many mathematicians and mathematical physicists there is taken a broad and inclusive view of modern mathematical physics. During the last century this science evolved within at least four components which illustrate [22] the development of the mathematics and quantum physics synergy:

(1) the use of ideas from mathematics in shedding new light on the existing principles of quantum physics, either from a conceptual or from a quantitative point of view;
(2) the use of ideas from mathematics in discovering new "laws of quantum physics";
(3) the use of ideas from quantum physics in shedding new light on existing mathematical structures;
(4) the use of ideas from quantum physics in discovering new domains in mathematics.

Each of these topics plays some role in understanding the modern mathematical physics. However, our success in directions (2) and (4) is certainly more modest than our success in directions (1) and (3). In some cases it is difficult to draw a clear-cut distinction between these two sets of components. In fact, we are lucky when it is possible to progress in directions (2) and (4), and when we make major progress there, historians like to speak of a revolution. In any case, many of mathematical physicists strive to understand within their research efforts these deep and lofty goals.

But there are many situations when mathematical physicists research efforts are directed toward another, more mundane aspect:

(5) the use of ideas from quantum physics and mathematics to benefit "economic competitiveness".

Here too, one might subdivide this aspect into conceptual understanding on the one hand (such as the mathematical model of Black and Sholes for pricing of derivative securities in financial markets) or invention on the other: the formulation of new algorithms or materials (e.g. quantum computers) which might revolutionize technology or change our way of life. As above, the boundary between these domains is not sharp, and it remains open to opinion and interpretation. The last strand can be characterized as "applied" mathematical physics. Rather we will restrict our analysis to the first four strands characterizing modern quantum physics and mathematics aspects. In fact, one believes that a case can be made that most of the profound applied directions arise after earlier fundamental quantum physics and mathematics progress.

We have passed through an extraordinary 35-year period of development of modern fundamental mathematics and quantum physics. Much of this development has drawn from one subject to understand other. Not only have concepts from diverse fields been united: statistical physics, quantum field theory and functional integration; gauge theory and geometry; index theory and knot invariants, etc., but also new phenomena have been recognized and new areas have emerged whose significance we only partially understand—both for mathematics and for modern quantum physics: such as noncommutative geometry, super-analysis, mirror symmetry, new topological invariants of manifolds, and the general notion of geometric quantization.

Over the past thirty five years, there is no question that the ideas from quantum physics have led to far greater invention of new mathematics, than the ideas from mathematics have been in discovering new laws of quantum physics. But this just underscores the opportunities of for future progress in the other direction. One very awaits a new understanding of the quantum nature of the world!

There has been great publicity and recognition attached to the progress made in modern geometry, in representation theory, and in deformation theory due to this interaction. But one should ignore the substantial deep progress in analysis and in probability theory, which unfortunately is more difficult to understand because of its delicate dependence on subtle notions of continuity.

On the other hand, there are deep differences between pure mathematics and modern quantum physics fundamentals. They have evolved from different cultures and they each have a distinctive set of values of their own, suited for their different realms of universality. But both subjects are strongly based on *intuition*, some natural and some acquired, which form our understanding. On the other hand, quantum physics describes the natural micro-world. hence physicists appeal to observation in order to verify the validity of a physical theory. And, although much of mathematics arises from the natural world, mathematics has no analogous testing ground—mathematicians appeal to their own set of values, namely mathematical proof, to justify validity of a mathematical theory. When in mathematical physics announcing results of a mathematical nature, it is necessary to claim a theorem when the proof meets the mathematical community standards for a proof, otherwise it is necessary to make a conjection with a detailed outline for support. Most of physics, on the other hand, has completely different standards.

There is no question that the interaction between modern mathematics and quantum physics will change radically during this running century. But we do hope that this evolution will preserve the positive experience of being a mathematician, of being a pure physicist, or of being a mathematical physicist, so that it remains attractive to the brightest and capable young students today and tomorrow.

It is instructive to look at the beginning of the XXth century and trace the way mathematics had been exerting the influence on the modern and classical quantum physics, and next observe the way the modern quantum physics is nowadays exerting so impressive influence on the modern mathematics. The latter will in part be a main topic of our present work, devoted to the application of the modern quantum mathematics to studying nonlinear dynamical systems in functional spaces. We will begin with a brief history of quantum mathematics:

The beginning of the XXth century:

- P.A.M. Dirac—first realized and used in quantum physics the fact that the commutator operation $D_a : \mathcal{A} \ni b \longrightarrow [a, b] \in \mathcal{A}$, where $a \in \mathcal{A}$ is fixed and $b \in \mathcal{A}$, is a differentiation of any associative algebra $\mathcal{A}$; moreover, he first constructed a spinor matrix realization of the Poincaré symmetry group $\mathcal{P}(1, 3)$ and invented the famous Dirac δ-function [9] (1920–1926);
- J. von Neumann—first applied the spectral theory of self-adjoint operators in Hilbert spaces to explain the radiation spectra of atoms and the related matter stability [32] (1926);
- V. Fock—first introduced the notion of many-particle Hilbert space, named by Fock space, and introduced the related creation and annihilation operators, acting in it [16] (1932);
- H. Weyl—first understood the fundamental role of the notion of symmetry in physics and developed a physics-oriented group theory; moreover he showed the importance of different representations of classical matrix groups for physics and studied the unitary representations of the Heisenberg-Weyl group related with creation and annihilation operators in Fock space [33] (1931).

The end of the XXth century. New developments are due to:

- L. Faddeev with co-workers—quantum inverse spectral theory transform [13] (1978);
- V. Drinfeld, S. Donaldson, E. Witten—quantum groups and algebras, quantum topology, quantum super-analysis [11, 12, 34] (1982–1994);
- Yu. Manin, R. Feynman—quantum information theory [14, 15, 28, 29] (1980–1986);
- P. Shor, E. Deutsch, L. Grover and others—quantum computer algorithms [8, 21, 31] (1985–1997).

As one can observe, many exciting and highly important mathematical achievements were strictly motivated by the impressive and deep influence of quantum physics ideas and ways of thinking, leading nowadays to an altogether new scientific field often called quantum mathematics.

Following this quantum mathematical way of thinking, we will demonstrate below that a wide class of strictly nonlinear dynamical systems in functional spaces can be treated as a natural object in specially constructed Fock spaces in which the corresponding evolution flows are completely linearized. Thereby, the powerful machinery of classical mathematical tools can be applied to studying the analytical properties of exact solutions to suitably well posed Cauchy problems.

2 Mathematical Preliminaries: Fock Space and Its Realizations

Let Φ be a separable Hilbert space, F be a topological real linear space and $\mathcal{A} := \{A(\varphi) : \varphi \in F\}$ a family of commuting self-adjoint operators in Φ (i.e. these operators commute in the sense of their resolutions of the identity). Consider the Gelfand rigging [2] of the Hilbert space Φ, i.e., a chain

$$\mathcal{D} \subset \Phi_+ \subset \Phi \subset \Phi_- \subset \mathcal{D}' \tag{2.1}$$

in which Φ_+ and Φ_- are further Hilbert spaces, and the inclusions are dense and continuous, i.e. Φ_+ is topologically (densely and continuously) and quasi-nuclearly (the inclusion operator $i : \Phi_+ \longrightarrow \Phi$ is of the Hilbert–Schmidt type) embedded into Φ, Φ_- is the dual of Φ_+ with respect to the scalar product $\langle .,. \rangle_\Phi$ in Φ, and $\mathcal{D}$ is a separable projective limit of Hilbert spaces, topologically embedded into Φ_+. Then, the following structural theorem [2, 3] holds:

Theorem 2.1 *Assume that the family of operators $\mathcal{A}$ satisfies the following conditions*:

(a) $\mathcal{D} \subset \operatorname{Dom} A(\varphi)$, $\varphi \in F$, *and the closure of the operator* $A(\varphi) \upharpoonright \mathcal{D}$ *coincides with* $A(\varphi)$ *for any* $\varphi \in F$, *that is* $\overline{A(\varphi) \upharpoonright \mathcal{D}} = A(\varphi)$ *in* Φ;
(b) *the Range* $A(\varphi) \upharpoonright \mathcal{D} \subset \Phi_+$ *for any* $\varphi \in F$;
(c) *for every* $f \in \mathcal{D}$ *the mapping* $F \ni \varphi \longrightarrow A(\varphi) f \in \Phi_+$ *is linear and continuous*;
(d) *there exists a strong cyclic* (*vacuum*) *vector* $|\Omega\rangle \in \bigcap_{\varphi \in F} \operatorname{Dom} A(\varphi)$, *such that the set of all vectors* $|\Omega\rangle$, $\prod_{j=1}^n A(\varphi_j)|\Omega\rangle$, $n \in \mathbb{Z}_+$, *is total in* Φ_+ (*i.e. their linear hull is dense in* Φ_+).

Then there exists a probability measure μ on $(F', C_\sigma(F'))$, where F' is the dual of F and $C_\sigma(F')$ is the σ-algebra generated by cylinder sets in F' such that, for μ-almost every $\eta \in F'$ there is a generalized joint eigenvector $\omega(\eta) \in \Phi_-$ of the family $\mathcal{A}$, corresponding to the joint eigenvalue $\eta \in F'$, that is

$$\langle \omega(\eta), A(\varphi) f \rangle_\Phi = \eta(\varphi) \langle \omega(\eta), f \rangle_\Phi \tag{2.2}$$

with $\eta(\varphi) \in \mathbb{R}$ denoting the pairing between F and F'.

The mapping

$$\Phi_+ \ni f \longrightarrow \langle \omega(\eta), f \rangle_\Phi := \hat{f}(\eta) \in \mathbb{C} \tag{2.3}$$

for any $\eta \in F'$ *can be continuously extended to a unitary surjective operator* $\mathcal{F} : \Phi \longrightarrow L_2^{(\mu)}(F'; \mathbb{C})$, *where*

$$\mathcal{F} f(\eta) := \hat{f}(\eta) \tag{2.4}$$

for any $\eta \in F'$ *is a generalized Fourier transform, corresponding to the family* $\mathcal{A}$. *Moreover, the image of the operator* $A(\varphi)$, $\varphi \in F'$, *under the* $\mathcal{F}$*-mapping is the operator of multiplication by the function* $F' \ni \eta \to \eta(\varphi) \in \mathbb{C}$.

We assume additionally that the main Hilbert space Φ possesses the standard Fock space (Bose)-structure [4, 6, 30], that is

$$\Phi = \bigoplus_{n \in \mathbb{Z}_+} \Phi_n, \tag{2.5}$$

where subspaces $\Phi_n := \Phi_{(s)}^{\otimes n}$, $n \in \mathbb{Z}_+$, are the symmetrized tensor products of a Hilbert space $\mathcal{H} := L_2(\mathbb{R}^m; \mathbb{C})$. If a vector $g := (g_0, g_1, \ldots, g_n, \ldots) \in \Phi$, its norm

$$\|g\|_\Phi := \left(\sum_{n \in \mathbb{Z}_+} \|g_n\|_n^2 \right)^{1/2}, \tag{2.6}$$

where $g_n \in \Phi_{(s)}^{\otimes n} \simeq L_{2,(s)}((\mathbb{R}^m)^n; \mathbb{C})$ and $\| \ldots \|_n$ is the corresponding norm in $\Phi_{(s)}^{\otimes n}$ for all $n \in \mathbb{Z}_+$. Denote here that, concerning the rigging structure (2.1), there holds the corresponding rigging for the Hilbert spaces $\Phi_{(s)}^{\otimes n}$, $n \in \mathbb{Z}_+$, that is

$$\mathcal{D}_{(s)}^n \subset \Phi_{(s),+}^{\otimes n} \subset \Phi_{(s)}^{\otimes n} \subset \Phi_{(s),-}^{\otimes n} \tag{2.7}$$

with some suitably chosen dense and separable topological spaces of symmetric functions $\mathcal{D}_{(s)}^n$, $n \in \mathbb{Z}_+$. Concerning expansion (2.5) we obtain by means of projective and inductive limits [2–4] the quasi-nucleus rigging of the Fock space Φ in the form (2.1):

$$\mathcal{D} \subset \Phi_+ \subset \Phi \subset \Phi_- \subset \mathcal{D}'.$$

Consider now any vector $|(\alpha)_n\rangle \in \Phi_{(s),}^{\otimes n}$ $n \in \mathbb{Z}_+$, which can be written [2, 6, 26] in the following canonical Dirac ket-form:

$$|(\alpha)_n\rangle := |\alpha_1, \alpha_2, \ldots, \alpha_n\rangle, \tag{2.8}$$

where, by definition,

$$|\alpha_1, \alpha_2, \ldots, \alpha_n\rangle := \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} |\alpha_{\sigma(1)}\rangle \otimes |\alpha_{\sigma(2)}\rangle \ldots |\alpha_{\sigma(n)}\rangle \tag{2.9}$$

and $|\alpha_j\rangle \in \Phi_{(s)}^{\otimes 1}(\mathbb{R}^m; \mathbb{C}) := \mathcal{H}$ for any fixed $j \in \mathbb{Z}_+$. The corresponding scalar product of base vectors as (2.9) is given as follows:

$$\begin{aligned} \langle (\beta)_n | (\alpha)_n \rangle &:= \langle \beta_n, \beta_{n-1}, \ldots, \beta_2, \beta_1 | \alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n \rangle \\ &= \sum_{\sigma \in S_n} \langle \beta_1 | \alpha_{\sigma(1)} \rangle \ldots \langle \beta_n | \alpha_{\sigma(n)} \rangle := per\{\langle \beta_i | \alpha_j \rangle : i, j = \overline{1, n}\}, \end{aligned} \tag{2.10}$$

where "*per*" denotes the permanent of matrix and $\langle .|.\rangle$ is the corresponding product in the Hilbert space $\mathcal{H}$. Based now on representation (2.8) one can define an operator $a^+(\alpha): \Phi_{(s)}^{\otimes n} \longrightarrow \Phi_{(s)}^{\otimes(n+1)}$ for any $|\alpha\rangle \in \mathcal{H}$ as follows:

$$a^+(\alpha)|\alpha_1, \alpha_2, \ldots, \alpha_n\rangle := |\alpha, \alpha_1, \alpha_2, \ldots, \alpha_n\rangle, \tag{2.11}$$

which is called the "creation" operator in the Fock space Φ. The adjoint operator $a(\beta) := (a^+(\beta))^* : \Phi_{(s)}^{\otimes(n+1)} \longrightarrow \Phi_{(s)}^{\otimes n}$ with respect to the Fock space Φ (2.5) for any $|\beta\rangle \in \mathcal{H}$, called the "annihilation" operator, acts as follows:

$$a(\beta)|\alpha_1, \alpha_2, \ldots, \alpha_{n+1}\rangle := \sum_{j=1}^{n+1} \langle \beta, \alpha_j\rangle |\alpha_1, \alpha_2, \ldots, \alpha_{j-1}, \hat{\alpha}_j, \alpha_{j+1}, \ldots, \alpha_{n+1}\rangle, \tag{2.12}$$

where the "*hat*" over a vector denotes that it should be omitted from the sequence.

It is easy to check that the commutator relationship

$$[a^+(\alpha), a(\beta)] = \langle \alpha, \beta\rangle \tag{2.13}$$

holds for any vectors $|\alpha\rangle \in \mathcal{H}$ and $|\beta\rangle \in \mathcal{H}$. Expression (2.13), owing to the rigged structure (2.1), can be naturally extended to the general case, when vectors $|\alpha\rangle$ and $|\beta\rangle \in \mathcal{H}_-$, conserving its form. In particular, if to take $|\alpha\rangle := |\alpha(x)\rangle = \frac{1}{\sqrt{2\pi}} e^{i\langle\lambda, x\rangle} \in \mathcal{H}_- := L_{2,-}(\mathbb{R}^m; \mathbb{C})$ for any $x \in \mathbb{R}^m$, one easily gets from (2.13) that

$$[a^+(x), a(y)] = \delta(x - y), \tag{2.14}$$

where we put, by definition, $a^+(x) := a^+(\alpha(x))$ and $a(y) := a(\alpha(y))$ for all $x, y \in \mathbb{R}^m$ and denoted by $\delta(\cdot)$ the classical Dirac delta-function.

The construction above makes it possible to observe easily that there exists a unique vacuum vector $|\Omega\rangle \in \mathcal{H}_+$, such that for any $x \in \mathbb{R}^m$

$$a(x)|\Omega\rangle = 0, \tag{2.15}$$

and the set of vectors

$$\left(\prod_{j=1}^{n} a^+(x_j)\right)|\Omega\rangle \in \Phi_{(s)}^{\otimes n} \tag{2.16}$$

is total in $\Phi_{(s)}^{\otimes n}$, that is their linear integral hull over the dual functional spaces $\hat{\Phi}_{(s)}^{\otimes n}$ is dense in the Hilbert space $\Phi_{(s)}^{\otimes n}$ for every $n \in \mathbb{Z}_+$. This means that for any vector $g \in \Phi$ the following representation

$$g = \bigoplus_{n\in\mathbb{Z}_+} \int_{(\mathbb{R}^m)^n} \hat{g}_n(x_1, \ldots, x_n) a^+(x_1) a^+(x_2) \cdots a^+(x_n)|\Omega\rangle \tag{2.17}$$

holds with the Fourier type coefficients $\hat{g}_n \in \hat{\Phi}_n := \hat{\Phi}_{(s)}^{\otimes n}$ for all $n \in \mathbb{Z}_+$, with $\hat{\Phi}_{(s)}^{\otimes 1} := \mathcal{H} \simeq L_2(\mathbb{R}^m; \mathbb{C})$. The latter is naturally endowed with the dual to (2.1) Gelfand type quasi-nucleus rigging

$$\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}, \tag{2.18}$$

making possible to construct a quasi-nucleus rigging of the dual Fock space $\hat{\Phi} := \bigoplus_{n\in\mathbb{Z}_+} \hat{\Phi}_n$. Thereby, chain (2.18) generates the dual Fock space quasi-nucleus rigging

$$\hat{\mathcal{D}} \subset \hat{\Phi}_+ \subset \hat{\Phi} \subset \hat{\Phi}_- \subset \hat{\mathcal{D}}' \tag{2.19}$$

with respect to the central Fock type Hilbert space $\hat{\Phi}$, where $\hat{\mathcal{D}} \simeq \mathcal{D}$, easily following from (2.1) and (2.18).

Construct now the following self-adjoint operator

$$a^+(x)a(x) := \rho(x) : \Phi \to \Phi, \tag{2.20}$$

called the density operator at a point $x \in \mathbb{R}^m$, satisfying the commutation properties:

$$\begin{aligned} &[\rho(x), \rho(y)] = 0, \\ &[\rho(x), a(y)] = -a(y)\delta(x-y), \\ &[\rho(x), a^+(y)] = a^+(y)\delta(x-y) \end{aligned} \tag{2.21}$$

for all $y \in \mathbb{R}^m$.

Now, if to construct the following self-adjoint family $\mathcal{A} := \{\int_{\mathbb{R}^m} \rho(x)\varphi(x)dx : \varphi \in F\}$ of linear operators in the Fock space Φ, where $F := \mathcal{S}(\mathbb{R}^m;\mathbb{R})$ is the Schwartz functional space, one can derive, making use of Theorem 2.1, that there exists the generalized Fourier transform (2.4), such that

$$\Phi(\mathcal{H}) = L_2^{(\mu)}(\mathcal{S}';\mathbb{C}) \simeq \int_{\mathcal{S}'}^{\oplus} \Phi_\eta d\mu(\eta) \tag{2.22}$$

for some Hilbert space sets Φ_η, $\eta \in F'$, and a suitable measure μ on $\mathcal{S}'$, with respect to which the corresponding joint eigenvector $\omega(\eta) \in \Phi_+$ for any $\eta \in F'$ generates the Fourier transformed family $\hat{\mathcal{A}} = \{\eta(\varphi) \in \mathbb{R} : \ \varphi \in \mathcal{S}\}$. Moreover, if $\dim \Phi_\eta = 1$ for all $\eta \in F$, the Fourier transformed eigenvector $\hat{\omega}(\eta) := \Omega(\eta) = 1$ for all $\eta \in F'$.

Now we will consider the family of self-adjoint operators $\mathcal{A}$ as generating a unitary family $\mathcal{U} := \{U(\varphi) : \varphi \in F\} = \exp(i\mathcal{A})$, where for any $\rho(\varphi) \in \mathcal{A}$, $\varphi \in F$, the operator

$$U(\varphi) := \exp[i\rho(\varphi)] \tag{2.23}$$

is unitary, satisfying the Abelian commutation condition

$$U(\varphi_1)U(\varphi_2) = U(\varphi_1 + \varphi_2) \tag{2.24}$$

for any $\varphi_1, \varphi_2 \in F$.

Since, in general, the unitary family $\mathcal{U} = \exp(i\mathcal{A})$ is defined in some Hilbert space Φ, being not necessarily of Fock type, the important problem of describing its Hilbertian cyclic representation spaces arises, within which the factorization

$$\rho(\varphi) = \int_{\mathbb{R}^m} a^+(x)a(x)\varphi(x)dx \tag{2.25}$$

jointly with relationships (2.21) hold for any $\varphi \in F$. This problem can be treated using mathematical tools devised both within the representation theory of C^*-algebras [10] and Gelfand–Vilenkin [17] approach. Below we will describe main features of the Gelfand–Vilenkin formalism, being much more suitable for the task, providing a reasonably unified framework of constructing the corresponding representations.

Definition 2.2 Let F be a locally convex topological vector space, $F_0 \subset F$ be a finite-dimensional subspace of F. Let $F^0 \subseteq F'$ be defined by

$$F^0 := \{\xi \in F' : \xi|_{F_0} = 0\}, \tag{2.26}$$

and called the annihilator of F_0.

The quotient space $F'^0 := F'/F^0$ may be identified with $F'_0 \subset F'$, the adjoint space of F_0.

Definition 2.3 Let $A \subseteq F'$; then the subset

$$X_{F^0}^{(A)} := \{\xi \in F' : \xi + F^0 \subset A\} \tag{2.27}$$

is called the cylinder set with base A and generating subspace F^0.

Definition 2.4 Let $n = \dim F_0 = \dim F'_0 = \dim F'^0$. One says that a cylinder set $X^{(A)}$ has Borel base, if A is Borel, when regarded as a subset of $\mathbb{R}^n$.

The family of cylinder sets with Borel base forms an algebra of sets.

Definition 2.5 The measurable sets in F' are the elements of the σ-algebra generated by the cylinder sets with Borel base.

Definition 2.6 A cylindrical measure in F' is a real-valued σ-pre-additive function μ defined on the algebra of cylinder sets with Borel base and satisfying the conditions $0 \le \mu(X) \le 1$ for any X, $\mu(F') = 1$ and $\mu(\coprod_{j\in\mathbb{Z}_+} X_j) = \sum_{j\in\mathbb{Z}_+} \mu(X_j)$, if all sets $X_j \subset F'$, $j \in \mathbb{Z}_+$, have a common generating subspace $F_0 \subset F$.

Definition 2.7 A cylindrical measure μ satisfies the commutativity condition if and only if for any bounded continuous function $\alpha : \mathbb{R}^n \longrightarrow \mathbb{R}$ of $n \in \mathbb{Z}_+$ real variables the function

$$\alpha[\varphi_1, \varphi_2, \ldots, \varphi_n] := \int_{F'} \alpha(\eta(\varphi_1), \eta(\varphi_2), \ldots, \eta(\varphi_n)) d\mu(\eta) \tag{2.28}$$

is sequentially continuous in $\varphi_j \in F$, $j = \overline{1, m}$. (It is well known [17, 18] that in countably normed spaces the properties of sequential and ordinary continuity are equivalent.)

Definition 2.8 A cylindrical measure μ is countably additive if and only if for any cylinder set $X = \coprod_{j\in\mathbb{Z}_+} X_j$, which is the union of countably many mutually disjoints cylinder sets $X_j \subset F'$, $j \in \mathbb{Z}_+$, $\mu(X) = \sum_{j\in\mathbb{Z}_+} \mu(X_j)$.

The following propositions hold.

Proposition 2.9 *A countably additive cylindrical measure μ can be extended to a countably additive measure on the σ-algebra, generated by the cylinder sets with Borel base. Such a measure will be also called a cylindrical measure.*

Proposition 2.10 *Let F be a nuclear space. Then any cylindrical measure μ on F', satisfying the continuity condition, is countably additive.*

Definition 2.11 Let μ be a cylindrical measure in F'. The Fourier transform of μ is the nonlinear functional

$$\mathcal{L}(\varphi) := \int_{F'} \exp[i\eta(\varphi)]d\mu(\eta). \tag{2.29}$$

Definition 2.12 The nonlinear functional $\mathcal{L} : F \longrightarrow \mathbb{C}$ on F, defined by (2.29), is called positive definite, if and only if for all $f_j \in F$ and $\lambda_j \in \mathbb{C}$, $j = \overline{1, n}$, the condition

$$\sum_{j,k=1}^{n} \bar{\lambda}_j \mathcal{L}(f_k - f_j)\lambda_k \geq 0 \tag{2.30}$$

holds for any $n \in \mathbb{Z}_+$.

Proposition 2.13 *The functional $\mathcal{L} : F \longrightarrow \mathbb{C}$ on F, defined by (2.29), is the Fourier transform of a cylindrical measure on F', if and only if it is positive definite, sequentially continuous and satisfying the condition $\mathcal{L}(0) = 1$.*

Suppose now that we have a continuous unitary representation of the unitary family $\mathcal{U}$ in a Hilbert space Φ with a cyclic vector $|\Omega\rangle \in \Phi$. Then we can put

$$\mathcal{L}(\varphi) := \langle\Omega|U(\varphi)|\Omega\rangle \tag{2.31}$$

for any $\varphi \in F := \mathcal{S}$, being the Schwartz space on $\mathbb{R}^m$, and observe that functional (2.31) is continuous on F owing to the continuity of the representation. Therefore, this functional is the generalized Fourier transform of a cylindrical measure μ on $\mathcal{S}'$:

$$\langle\Omega|U(\varphi)|\Omega\rangle = \int_{\mathcal{S}'} \exp[i\eta(\varphi)]d\mu(\eta). \tag{2.32}$$

From the spectral point of view, based on Theorem 2.1 there is an isomorphism between the Hilbert spaces Φ and $L_2^{(\mu)}(\mathcal{S}'; \mathbb{C})$, defined by $|\Omega\rangle \longrightarrow \Omega(\eta) = 1$ and $U(\varphi)|\Omega\rangle \longrightarrow \exp[i\eta(\varphi)]$ and next extended by linearity upon the whole Hilbert space Φ.

In the case of the non-cyclic case there exists a finite or countably infinite family of measures $\{\mu_k : k \in \mathbb{Z}_+\}$ on $\mathcal{S}'$, with $\Phi \simeq \bigoplus_{k\in\mathbb{Z}_+} L_2^{(\mu_k)}(\mathcal{S}'; \mathbb{C})$ and the unitary operator $U(\varphi) : \Phi \longrightarrow \Phi$ for any $\varphi \in \mathcal{S}'$ corresponds in all $L_2^{(\mu_k)}(\mathcal{S}'; \mathbb{C})$, $k \in \mathbb{Z}_+$, to $\exp[i\eta(\varphi)]$. This means that there exists a single cylindrical measure μ on $\mathcal{S}'$ and a μ-measurable field of Hilbert spaces Φ_η on $\mathcal{S}'$, such that

$$\Phi \simeq \int_{\mathcal{S}'}^{\oplus} \Phi_\eta d\mu(\eta), \tag{2.33}$$

with $U(\varphi) : \Phi \longrightarrow \Phi$, corresponding [17] to the operator of multiplication by $\exp[i\eta(\varphi)]$ for any $\varphi \in \mathcal{S}$ and $\eta \in \mathcal{S}'$. Thereby, having constructed the nonlinear functional (2.29) in an exact analytical form, one can retrieve the representation of the unitary family $\mathcal{U}$ in the corresponding Hilbert space Φ of the Fock type, making use of the suitable factorization (2.25) as follows: $\Phi = \oplus_{n\in\mathbb{Z}_+} \Phi_n$, where

$$\Phi_n = \underset{f_n \in L_{2,s}((R^m)^n;\mathbb{C})}{span} \left\{ \prod_{j=\overline{1,n}} a^+(x_j)|\Omega\rangle \right\}, \tag{2.34}$$

for all $n \in \mathbb{Z}_+$. The cyclic vector $|\Omega\rangle \in \Phi$ can be, in particular, obtained as the ground state vector of some unbounded self-adjoint positive definite Hamilton operator $\mathbb{H} : \Phi \longrightarrow \Phi$, commuting with the self-adjoint particles number operator

$$\mathbb{N} := \int_{\mathbb{R}^m} \rho(x)dx, \tag{2.35}$$

that is $[\mathbb{H}, \mathbb{N}] = 0$. Moreover, the conditions

$$\mathbb{H}|\Omega\rangle = 0 \tag{2.36}$$

and

$$\inf_{g \in \operatorname{dom} \mathbb{H}} \langle g, \mathbb{H} g\rangle = \langle \Omega|\mathbb{H}|\Omega\rangle = 0 \tag{2.37}$$

hold for the operator $\mathbb{H} : \Phi \longrightarrow \Phi$, where $\operatorname{dom} \mathbb{H}$ denotes its domain of definition.

To find the functional (2.31), which is called the generating Bogolubov type functional for moment distribution functions

$$F_n(x_1, x_2, \ldots, x_n) := \langle \Omega| : \rho(x_1)\rho(x_2)\ldots\rho(x_n) : |\Omega\rangle, \tag{2.38}$$

where $x_j \in \mathbb{R}^m$, $j = \overline{1, n}$, and the normal ordering operation $: \cdot :$ is defined as

$$: \rho(x_1)\rho(x_2)\ldots\rho(x_n) : = \prod_{j=1}^{n} \left(\rho(x_j) - \sum_{k=1}^{j} \delta(x_j - x_k)\right), \tag{2.39}$$

it is convenient to choose the Hamilton operator $\mathbb{H} : \Phi \longrightarrow \Phi$ in the following [7, 18, 19] algebraic form:

$$\mathbb{H} := \frac{1}{2}\int_{\mathbb{R}^m} K^+(x)\rho^{-1}(x)K(x)dx + V(\rho), \tag{2.40}$$

being equivalent in the Hilbert space Φ to the positive definite operator expression

$$\mathbb{H} := \frac{1}{2}\int_{\mathbb{R}^m} (K^+(x) - A(x;\rho))\rho^{-1}(x)(K(x) - A(x;\rho))dx, \tag{2.41}$$

where $A(x;\rho) : \Phi \to \Phi,\ x \in \mathbb{R}^m$, is some specially chosen linear self-adjoint operator. The "potential" operator $V(\rho) : \Phi \longrightarrow \Phi$ is, in general, a polynomial (or analytical) functional of the density operator $\rho(x) : \Phi \longrightarrow \Phi$ and the operator is given as

$$K(x) := \nabla_x \rho(x)/2 + iJ(x), \tag{2.42}$$

where the self-adjoint "current" operator $J(x) : \Phi \longrightarrow \Phi$ can be defined (but non-uniquely) from the equality

$$\partial\rho/\partial t = \frac{1}{i}[\mathbb{H}, \rho(x)] = -\langle \nabla_x \cdot J(x)\rangle, \tag{2.43}$$

holding for all $x \in \mathbb{R}^m$. Such an operator $J(x) : \Phi \longrightarrow \Phi$, $x \in \mathbb{R}^m$ can exist owing to the commutation condition $[\mathbb{H}, \mathbb{N}] = 0$, giving rise to the continuity relationship (2.43), if to take into account that supports $\operatorname{supp} \rho$ of the density operator $\rho(x) : \Phi \longrightarrow \Phi$, $x \in \mathbb{R}^m$, can be chosen arbitrarily owing to the independence of (2.43) on the potential operator

$V(\rho):\Phi\longrightarrow\Phi$, but its strict dependence on it of the corresponding representation (2.33). Denote also that representation (2.41) holds only under the condition that there exists such a self-adjoint operator $A(x;\rho):\Phi\longrightarrow\Phi$, $x\in\mathbb{R}^m$, that

$$K(x)|\Omega\rangle = A(x;\rho)|\Omega\rangle \tag{2.44}$$

for all ground states $|\Omega\rangle\in\Phi$, corresponding to suitably chosen potential operators $V(\rho):\Phi\longrightarrow\Phi$.

The self-adjointness of the operator $A(x;\rho):\Phi\longrightarrow\Phi$, $x\in\mathbb{R}^m$, can be stated following schemes from works [7, 19], under the additional condition of the existence of such a linear anti-unitary mapping $T:\Phi\longrightarrow\Phi$ that the following invariance conditions hold:

$$T\rho(x)T^{-1}=\rho(x),\qquad TJ(x)T^{-1}=-J(x),\qquad T|\Omega\rangle=|\Omega\rangle \tag{2.45}$$

for any $x\in\mathbb{R}^m$. Thereby, owing to conditions (2.45), the following expressions

$$K^*(x)|\Omega\rangle = A(x;\rho)|\Omega\rangle = K(x)|\Omega\rangle \tag{2.46}$$

hold for any $x\in\mathbb{R}^m$, giving rise to the self-adjointness of the operator $A(x;\rho):\Phi\longrightarrow\Phi$, $x\in\mathbb{R}^m$.

Based now on the construction above one deduces easily from expression (2.43) that the generating Bogolubov type functional (2.31) obeys for all $x\in\mathbb{R}^m$ the following functional-differential equation:

$$[\nabla_x - i\nabla_x\varphi]\frac{1}{2i}\frac{\delta\mathcal{L}(\varphi)}{\delta\varphi(x)} = A\left(x;\frac{1}{i}\frac{\delta}{\delta\varphi}\right)\mathcal{L}(\varphi), \tag{2.47}$$

whose solutions should satisfy the Fourier transform representation (2.32). In particular, a wide class of special so-called Poissonian white noise type solutions to the functional-differential equation (2.47) was obtained in [5, 7, 19] by means of functional-operator methods in the following generalized form:

$$\mathcal{L}(\varphi)=\exp\left\{A\left(\frac{1}{i}\frac{\delta}{\delta\varphi}\right)\right\}\exp\left(\bar\rho\int_{\mathbb{R}^m}\{\exp[i\varphi(x)]-1\}dx\right), \tag{2.48}$$

where $\bar\rho:=\langle\Omega|\rho|\Omega\rangle\in\mathbb{R}_+$ is a Poisson distribution density parameter.

Consider now the case, when the basic Fock space $\Phi=\bigotimes_{j=1}^s\Phi^{(j)}$, where $\Phi^{(j)}$, $j=\overline{1,s}$, are Fock spaces corresponding to the different types of independent cyclic vectors $|\Omega_j\rangle\in\Phi^{(j)}$, $j=\overline{1,s}$. This, in particular, means that the suitably constructed creation and annihilation operators $a_j(x), a_k^+(y):\Phi\longrightarrow\Phi$, $j,k=\overline{1,s}$, satisfy the following commutation relations:

$$\begin{aligned}&[a_j(x),a_k(y)]=0,\\&[a_j(x),a_k^+(y)]=\delta_{jk}\delta(x-y)\end{aligned} \tag{2.49}$$

for any $x,y\in\mathbb{R}^m$.

Definition 2.14 A vector $|u\rangle\in\Phi$, $x\in\mathbb{R}^m$, is called coherent with respect to a mapping $u\in L_2(\mathbb{R}^m;\mathbb{R}^s):=M$, if it satisfies the eigenfunction condition

$$a_j(x)|u\rangle = u_j(x)|u\rangle \tag{2.50}$$

for each $j=\overline{1,s}$ and all $x\in\mathbb{R}^m$.

It is easy to check that the coherent vectors $|u\rangle \in \Phi$ exist. Really, the following vector expression

$$|u\rangle := \exp\{(u, a^{+})\}|\Omega\rangle, \tag{2.51}$$

where $(.,.)$ is the standard scalar product in the Hilbert space M, satisfies the defining condition (2.50), and moreover, the norm

$$\|u\|_{\Phi} := \langle u|u\rangle^{1/2} = \exp\left(\frac{1}{2}\|u\|^2\right) < \infty, \tag{2.52}$$

since $u \in M$ and its norm $\|u\| := (u, u)^{1/2}$ is bounded.

3 The Fock Space Embedding Method, Nonlinear Dynamical Systems and Their Complete Linearization

Consider any function $u \in M := L_2(\mathbb{R}^m; \mathbb{R}^s)$ and observe that the Fock space embedding mapping

$$\xi : M \ni u \longrightarrow |u\rangle \in \Phi, \tag{3.1}$$

defined by means of the coherent vector expression (2.51) realizes a smooth isomorphism between Hilbert spaces M and Φ. The inverse mapping $\xi^{-1} : \Phi \longrightarrow M$ is given by the following exact expression:

$$u(x) = \langle\Omega|a(x)|u\rangle, \tag{3.2}$$

holding for almost all $x \in \mathbb{R}^m$. Owing to condition (2.52) one finds from (3.2) that the corresponding function $u \in M$.

Let now in the Hilbert space M be defined a nonlinear dynamical system (which can be, in general, non-autonomous) in partial derivatives

$$du/dt = K[u], \tag{3.3}$$

where $t \in \mathbb{R}_+$ is the corresponding evolution parameter, $[u] := (t, x; u, u_x, u_{xx}, \ldots, u_{rx})$, $r \in \mathbb{Z}_+$, and a mapping $K : M \longrightarrow T(M)$ is Frechet smooth. Assume also that the Cauchy problem

$$u|_{t=+0} = u_0 \tag{3.4}$$

is solvable for any $u_0 \in M$ in an interval $[0, T) \subset \mathbb{R}^1_+$ for some $T > 0$. Thereby, there is defined the smooth evolution mapping

$$T_t : M \ni u_0 \longrightarrow u(t|u_0) \in M, \tag{3.5}$$

for all $t \in [0, T)$.

It is now natural to consider the following commuting diagram

$$\begin{array}{ccc} M & \stackrel{\xi}{\longrightarrow} & \Phi \\ T_t \downarrow & & \downarrow \mathbb{T}_t \\ M & \stackrel{\xi}{\longrightarrow} & \Phi, \end{array} \tag{3.6}$$

where the mapping $\mathbb{T}_t : \Phi \longrightarrow \Phi$, $t \in [0, T)$, is defined from the conjugation relationship

$$\xi \circ T_t = \mathbb{T}_t. \circ \xi. \tag{3.7}$$

Take now the corresponding to $u_0 \in M$ coherent vector $|u_0\rangle \in \Phi$ and construct the vector

$$|u\rangle := \mathbb{T}_t \cdot |u_0\rangle \tag{3.8}$$

for all $t \in [0, T)$. Since vector (3.8) is, by construction, coherent, that is

$$a_j(x)|u\rangle := u_j(x, t|u_0)|u\rangle \tag{3.9}$$

for each $j = \overline{1, s}$, $t \in [0, T)$ and almost all $x \in \mathbb{R}^m$, owing to the smoothness of the mapping $\xi : M \longrightarrow \Phi$ with respect to the corresponding norms in the Hilbert spaces M and Φ, we derive that coherent vector (3.8) is differentiable with respect to the evolution parameter $t \in [0, T)$. Thus, one can find easily [24, 26] that

$$\frac{d}{dt}|u\rangle = \hat{K}[a^+, a]|u\rangle, \tag{3.10}$$

where

$$|u\rangle|_{t=+0} = |u_0\rangle \tag{3.11}$$

and a mapping $\hat{K}[a^+, a] : \Phi \longrightarrow \Phi$ is defined by the exact analytical expression

$$\hat{K}[a^+, a] := (a^+, K[a]). \tag{3.12}$$

As a result of the consideration above we obtain the following theorem.

Theorem 3.1 *Any smooth nonlinear dynamical system (3.3) in Hilbert space $M := L_2(\mathbb{R}^m; \mathbb{R}^s)$ is representable by means of the Fock space embedding isomorphism $\xi : M \longrightarrow \Phi$ in the completely linear form (3.10).*

Make now some comments concerning the solution to the linear equation (3.10) under the Cauchy condition (3.11). Since any vector $|u\rangle \in \Phi$ allows the series representation

$$\begin{aligned} |u\rangle = &\bigoplus_{n:=\sum_{j=1}^s n_j \in \mathbb{Z}_+} \frac{1}{(n_1! n_2! \dots n_s!)^{1/2}} \\ &\times \int_{(\mathbb{R}^m)^n} f^{(n)}_{n_1 n_2 \dots n_s}\left(x_1^{(1)}, x_2^{(1)}, \dots, x_{n_1}^{(1)}; x_1^{(2)}, x_2^{(2)}, \dots, x_{n_2}^{(2)}; \dots; x_1^{(s)}, x_2^{(s)}, \dots, x_{n_s}^{(s)}\right) \\ &\times \prod_{j=1}^s \left(\prod_{k=1}^{n_j} dx_k^{(j)} a_j^+(x_k^{(j)}) \right) |\Omega\rangle, \end{aligned} \tag{3.13}$$

where for any $n = \sum_{j=1}^s n_j \in \mathbb{Z}_+$ functions

$$f^{(n)}_{n_1 n_2 \dots n_s} \in \bigotimes_{j=1}^s L_{2,s}((\mathbb{R}^m)^{n_j}; \mathbb{C}) \simeq L_{2,s}\left(\mathbb{R}^{mn_1} \times \mathbb{R}^{mn_2} \times \cdots \mathbb{R}^{mn_s}; \mathbb{C}\right), \tag{3.14}$$

and the norm

$$\|u\|_{\Phi}^2 = \sum_{n=\sum_{j=1}^{s} n_j} \|f_{n_1 n_2 \ldots n_s}^{(n)}\|_2^2 = \exp(\|u\|^2). \tag{3.15}$$

Equation (3.10), by substitution (3.13) into it, reduces to an infinite recurrent set of linear evolution equations in partial derivatives on coefficient functions (3.14). The latter can often be solved [24] step by step analytically in exact form, thereby, making it possible to obtain owing to representation (3.2), the exact solution $u \in M$ to the Cauchy problem (3.4) for our nonlinear dynamical system in partial derivatives (3.3).

Remark 3.2 Concerning some applications of nonlinear dynamical systems like (3.1) in mathematical physics problems, it is very important to construct their so called conservation laws or smooth invariant functionals $\gamma : M \longrightarrow \mathbb{R}$ on M. Making use of the quantum mathematics technique described above one can suggest an effective algorithm for construction these conservation laws in exact form.

Really, consider a vector $|\gamma\rangle \in \Phi$, satisfying the linear equation:

$$\frac{\partial}{\partial t}|\gamma\rangle + \hat{K}^*[a^+, a]|\gamma\rangle = 0. \tag{3.16}$$

Then the following proposition [24] holds.

Proposition 3.3 *The functional*

$$\gamma := \langle u|\gamma\rangle \tag{3.17}$$

is a conservation law for dynamical system (3.1), *that is*

$$d\gamma/dt|_K = 0 \tag{3.18}$$

along any orbit of the evolution mapping (3.5).

4 Conclusion

Within the scope of this work we have described main mathematical preliminaries and properties of the quantum mathematics techniques suitable for analytical studying the important linearization problem for a wide class of nonlinear dynamical systems in partial derivatives in Hilbert spaces. This problem was analyzed in many details using the Gelfand–Vilenkin representation theory [17] of infinite-dimensional groups and the Goldin–Menikoff–Sharp theory [18–20] of generating Bogolubov type functionals, classifying these representations. The related problem of constructing Fock type space representations and retrieving their creation-annihilation generating structure still needs a deeper investigation within the approach devised. We here mention only that some aspects of this problem within the so-called Poissonian White noise analysis were studied in a series of works [1, 2, 23, 27], based on some generalization of the Delsarte type characters technique. It is necessary to mention also the related results obtained in [24–26], devoted to application of the Fock space embedding method to finding conservation laws and the so called recursion operators for the well known Korteweg–de Vries type nonlinear dynamical systems. Concerning some of important applications of the methods devised in the work to concrete dynamical systems, we plan to devote next investigation.

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References

1. Albeverio, S., Kondratiev, Y.G., Streit, L.: How to generalize white noise analysis to non-Gaussian measures. Preprint Bi Bo S, Bielefeld (1992)
2. Berezansky, Y.M.: A generalization of white noise analysis by means of theory of hypergroups. Rep. Math. Phys. **38**(3), 289–300 (1996)
3. Berezansky, Y.M., Kondratiev, Y.G.: Spectral Methods in Infinite-Dimensional Analysis, vols. 1, 2. Kluwer, Dordrecht (1995)
4. Berezin, F.A.: The Second Quantization Method. Nauka, Moscow (1986) (in Russian)
5. Bogoliubov, N.N.: Collected Works, vol. 2. Naukova Dumka, Kiev (1960) (in Russian)
6. Bogolubov, N.N., Bogolubov, N.N. Jr.: Introduction into Quantum Statistical Mechanics. World Scientific, Dordrecht (1986), p. 384
7. Bogoliubov, N.N. Jr., Prykarpatsky, A.K.: Quantum method of generating Bogolubov functionals in statistical physics: current Lie algebras, their representations and functional equations. Phys. Elementary Part. At. Nucl. **17**(4), 791–827 (1986) (in Russian)
8. Deutsch, D.: Quantum theory, the Church-Turing principle and the universal quantum computer. Proc. Roy. Soc. Lond. A **400**, 97–117 (1985)
9. Dirac, P.A.M.: The Principles of Quantum Mechanics. Oxford University Press, Oxford (1932)
10. Dixmier, J.: C*-Algebras. North-Holland, Amsterdam (1982)
11. Drinfeld, V.G.: Quantum groups. In: Proc. of the Int. Congress of Mathematicians, pp. 798–820 (1986)
12. Donaldson, S.K.: An application of gauge theory to four-dimensional topology. J. Differ. Geom. **17**, 279–315 (1982)
13. Faddeev, L.D., Sklyanin, E.K.: Quantum mechanical approach to completely integrable field theories. Proc. USSR Acad. Sci. (DAN) **243**, 1430–1433 (1978) (in Russian)
14. Feynman, R.: Simulating physics with computers. Int. J. Theor. Phys. **21**, 467–488 (1982)
15. Feynman, R.: Quantum mechanical computers. Found. Phys. **16**, 507–531 (1986)
16. Fock, V.A.: Konfigurationsraum und zweite Quantelung. Zeischrift Phys. **75**, 622–647 (1932)
17. Gelfand, I., Vilenkin, N.: Generalized Functions, vol. 4. Academic Press, New York (1964)
18. Goldin, G.A.: Nonrelativistic current algebras as unitary representations of groups. J. Math. Phys. **12**(3), 462–487 (1971)
19. Goldin, G.A., Grodnik, J., Powers, R.T., Sharp, D.: Nonrelativistic current algebra in the N/V limit. J. Math. Phys. **15**, 88–100 (1974)
20. Goldin, G.A., Menikoff, R., Sharp, F.H.: Diffeomorphism groups, gauge groups, and quantum theory. Phys. Rev. Lett. **51**, 2246–2249 (1983)
21. Grover, L.K.: Quantum mechanics helps in searching for a needle in a haystack. Phys. Rev. Lett. **79**, 325–328 (1997)
22. Jaffe, A., Quinn, F.: Theoretical mathematics: toward a cultural synthesis of mathematics and theoretical physics. Bull. Am. Math. Soc. **29**, 1–13 (1993); Zeischrift Phys. **75** 622–647 (1932)
23. Kondratiev, Y.G., Streit, L., Westerkamp, W., Yan, J.-A.: Generalized functions in infinite-dimensional analysis II. AS Preprint (1995)
24. Kowalski, K.: Methods of Hilbert Spaces in the Theory of Nonlinear Dynamical Systems. World Scientific, Singapore (1994)
25. Kowalski, K., Steeb, W.-H.: Symmetries and first integrals for nonlinear dynamical systems: Hilbert space approach. I, II. Progr. Theor. Phys. **85**(4), 713–722 (1991), **85**(4) 975–983 (1991)
26. Kowalski, K., Steeb, W.-H.: Non-linear Dynamical Systems and Carleman Linearization. World Scientific, Singapore (1991)
27. Lytvynov, E.W., Rebenko, A.L., Shchepaniuk, G.V.: Wick calculus on spaces of generalized functions compound Poisson white noise. Rep. Math. Phys. **39**(2), 219–247 (1997)
28. Manin, Y.I.: Computable and Uncomputable. Sov. Radio, Moscow (1980) (in Russian)
29. Manin, Y.I.: Classical computation, quantum computation and P. Shor's factorizing algorithm. Proc. Bourbaki Seminar (1999)

30. Prykarpatsky, A.K., Taneri, U., Bogolubov, N.N., Jr.: Quantum Field Theory and Application to Quantum Nonlinear Optics. World Scientific, New York (2002)
31. Shor, P.W.: Polynomial time algorithms for prime factorization and discrete logarithms on a quantum computer. SIAM J. Comput. **26**(5), 1484–1509 (1997)
32. Von Neumann, J.: Mathematische Grundlagen der Quanten Mechanik. Springer, Berlin (1932)
33. Weyl, H.: The Theory of Groups and Quantum Mechanics. Dover, New York (1931)
34. Witten, E.: Non-abelian bozonization in two dimensions. Commun. Math. Phys. **92**, 455–472 (1984)